Quiz 4 (*Oct. 26, Marks: 4, Duration: 30 mins*) **COL703, Jul-Nov 2023**

Name: **Entry No.:**

1. **[1 marks]** Write down a formula in first order logic with equality that captures the requirement that a unary predicate P holds for exactly three different elements.

2. Consider the following structures over a signature with a single binary relation symbol R.

$$U_{\mathcal{A}} = \mathbb{N} \text{ and } R_{\mathcal{A}} = \{(n, m) \in \mathbb{N} \times \mathbb{N} \quad \text{such that} \quad n < m\}$$
$$U_{\mathcal{B}} = \mathbb{Z} \text{ and } R_{\mathcal{B}} = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \quad \text{such that} \quad n < m\}$$
$$U_{\mathcal{C}} = \mathbb{Q} \text{ and } R_{\mathcal{C}} = \{(n, m) \in \mathbb{Q} \times \mathbb{Q} \quad \text{such that} \quad n < m\}$$

   - **[0.5 marks]** Give a predicate logic sentence (i.e., formula with no free variables) that is satisfied by $\mathcal{B}$ but not by $\mathcal{A}$.
   - **[0.5 marks]** Give a predicate logic sentence (i.e., formula with no free variables) that is satisfied by $\mathcal{C}$ but not by $\mathcal{B}$.

3. **[1 marks]** Recall the statement of Herbrand's theorem: Let $F = \forall x_1 \forall x_2 \ldots \forall x_n \; F^*$ be a closed formula in Skolem form. Then F is satisfiable if and only if it has a Herbrand model.

   We claim that this is not true when F is an arbitrary formula. Consider, $S = P(a) \wedge \exists x \; \neg P(x)$ over a signature that consists of a constant symbol a and a predicate symbol P.

   Argue that S has a model, but it has no Herbrand models.

4. **[1 marks]** Consider the following predicate-logic sentences.

   ϕ_1: $\forall x \; P(x, x)$
   ϕ_2: $\forall x \forall y \; (P(x, y) \rightarrow P(y, x))$
   ϕ_3: $\forall x \forall y \forall z \; (P(x, y) \wedge P(y, z) \rightarrow P(x, z))$

   These sentences express that P is reflexive, symmetric, and transitive.

   Show that transitivity is not semantically entailed by the other two properties. In other words, give a model (an assignment) that satisfies ϕ_1 and ϕ_2, but does not satisfy ϕ_3.