Uniqueness of ROBDD of a given boolean function

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Claim: For a function $f : \{0,1\}^n \rightarrow \{0,1\}$ there is exactly one ROBDD u with ordering $x_1 < x_2 < \ldots < x_n$ such that u represents $f(x_1, x_2, \ldots, x_n)$.

Proof: By induction on the number of parameters.

The *base case* ($n = 0$) is trivial. There are two function with 0 parameters – f() = 0 and f() = 1 – the ROBDDs for which are just the terminal nodes $\boxed{0}$ and $\boxed{1}$.

For the *inductive step*, we assume that functions with $n-1$ parameters have a unique ROBDD.

Consider functions f_0 and f_1 defined as:

$$f_0(x_2, x_3, \ldots, x_n) = f(0, x_2, x_3, \ldots, x_n), \text{ and}$$
$$f_1(x_2, x_3, \ldots, x_n) = f(1, x_2, x_3, \ldots, x_n) \qquad .$$

From the induction hypothesis: there are unique ROBDDs u_0 and u_1 representing f_0 and f_1 respectively.

If $\underline{u_0 = u_1}$, then $f = f_0 = f_1$. And therefore u_0 (or, u_1) represents f. If there was a $u' \neq u_0$ that also represented f, we claim that u' must have x_1 as the root. (Why? Because if x_1 does not appear at the root, it cannot appear anywhere else because of the given ordering. And if it does not appear at all, then f is same as f_0 and f_1, and therefore, $u' \neq u_0$ is also a ROBDD for f_0 and f_1, which is against the induction hypothesis.)

So, we agree that x_1 must be the root of u'. The dotted edge from x_1 must lead to u_0 and the solid edge must lead to u_1. But since u_0 and u_1 are equal u' cannot be a reduced OBDD!

Now, suppose $\underline{u_0 \neq u_1}$. We claim that any ROBDD for f must have x_1 as the root. (Why? Because if x_1 does not appear at the root, it cannot appear anywhere else because of the given ordering. And if it does not appear at all, then f is same as f_0 and f_1. Which means that u_0 and u_1 are different ROBDDs for the same function! This is against our induction hypothesis.)

Now, let us say that u' and u'' are two different ROBDDs for f. But we know that both u' and u'' must have x_1 as the root. Moreover, x_1 on a dotted-edge must go to a node that computes f_0. So, the ROBDD rooted at that node must be the same as u_0 (induction hypothesis; there cannot be two ROBDDs for f_0). Also, x_1 on a solid-edge must go to a node that computes f_1. So, the ROBDD rooted at that node must be the same as u_1 (induction hypothesis, once again).

So, we know that u' and u'' have the same root, and the ROBDDs rooted at the root's low-child is the same for u' and u''. Also, the ROBDDs rooted at the root's high-child is the same for u' and u''. We claim that if u' and u'' are not the same, it must be because of a shared node in u' (say, without loss of generality) such that u'' has two copies of that node. (If the difference is for any other reason, the ROBDDs rooted at the the low-child, or the high-child, of the root cannot be identical.) But if there are two copies of a node in u'', it cannot be reduced! Hence, u' and u'' must be the same.

Existence of an ROBBD for f is simple – we can simply take x_1 as the root, put f_0 as dotted-edge child, put f_1 as solid-edge child, and reduce.