COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 1-2 (Propositional Logic, Natural Deduction Proofs)

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Motivation

- The aim of logic in computer science is to develop languages to model the situations we encounter.
- Why? So that we reason about the situations formally.
- Why do we want to do this formally? So that we make <u>valid</u> arguments while reasoning, so that the arguments can be defended rigorously, and can even be executed on a machine.

Example

- If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late. *Therefore*, there were taxis at the station.

- Intuitively, the argument seems valid. Why?

- The sentence after the 'therefore' logically follows from the sentences before it.

Another example

- If it is raining and Mark does not have his umbrella with him, then he will get wet. Mark is not wet. It is raining. *Therefore*, Mark has his umbrella with him.
- Seems a valid argument as well. In fact, has the same structure as the previous example.

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- If p and not q, then r. Not r. p. *Therefore*, q.

Declarative sentences

- In order to make arguments rigorous, we need to develop a language.
- To express sentences in a way that brings out their logical structure.
- Propositional logic – based on propositions (or declarative sentences) which one can argue as being true or false.

Declarative sentences

- The sum of numbers 3 and 5 equals 8.
- All students registered for COL703 are present in the class today.
- We won't consider non-declarative sentences: e.g. Could you please pass me the salt?
- We are interested in precise declarative sentences, or statements about the behaviour of computer systems, or programs.
- And a calculus of reasoning (so that we can argue whether certain inferences can be made correctly or not).

Propositional logic

- Consider atomic, or indecomposable, declarative sentences, e.g., "The number 5 is even."
- We assign distinct symbols to these atomic sentences: p, q, r, ...
- Code up complex sentences in a compositional way, using symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$.

Example

- p: The number 5 is even.
- q: The number 5 is prime.
- r: The number 5 is bigger than the number 2.

- $\neg p$: The number 5 is **not** even. (Or, equivalently, it is **not** true that the number 5 is even.)
- $p \vee q$: (at least one of these is true) The number 5 is even **or** the number 5 is prime.
- $p \wedge q$: The number 5 is even **and** the number 5 is prime.
- $q \rightarrow p$: **If** the number 5 is prime, **then** the number 5 is even.

Binding priorities

- $\neg$ binds more tightly than $\wedge$ and $\vee$
- $\wedge$ and $\vee$ bind more tightly than $\rightarrow$
- $\rightarrow$ is right-associative: $p \rightarrow q \rightarrow r$ denotes $p \rightarrow (q \rightarrow r)$
- $p \wedge q \rightarrow \neg r \vee q \quad$ is $\quad (p \wedge q) \rightarrow ((\neg r) \vee q)$

Natural deduction

- Calculus for reasoning about propositions.
- Proof rules that can allow us to infer formulas from other formulas.
- $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

  (e.g., $p \wedge \neg q \to r$, $\neg r$, $p \vdash q$)
- This sequent is valid if a proof for it can be found using the proof rules.
- The rules should allow valid arguments and disallow invalid ones.

Rules for natural deduction

1. Rules for conjunction

Examples

Prove that $p \wedge q, r \vdash q \wedge r$ is valid.

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Prove that $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$ is valid.

Rules for natural deduction

1. Rules for conjunction
2. Rules for double negation

Examples

Prove that $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid.

Rules for natural deduction

1. Rules for conjunction
2. Rules for double negation
3. Rules for implication

Examples

Prove that $p, p \to q, p \to (q \to r) \vdash r$ is valid. (elimination)

Prove that $p \to (q \to r), p, \neg r \vdash \neg q$ is valid. (elimination)

Prove that $p \to q \vdash \neg q \to \neg p$ is valid. (introduction)

Prove that $\neg q \to \neg p \vdash p \to \neg\neg q$ is valid. (introduction)

Examples (Theorems)

Prove that $\vdash p \to p$ is valid.

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Prove that $\vdash p \to p$ is valid.

Prove that $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$ is valid.

Some more examples

$p \wedge q \to r \vdash p \to (q \to r)$

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$p \to q \vdash p \wedge r \to q \wedge r$

Rules for natural deduction

1. Rules for conjunction
2. Rules for double negation
3. Rules for implication
4. Rules for disjunction

Examples

$p \vee q \vdash q \vee p$

$q \to r \vdash p \vee q \to p \vee r$

$p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

Example

$p \rightarrow (q \rightarrow p)$

Rules for natural deduction

1. Rules for conjunction
2. Rules for double negation
3. Rules for implication
4. Rules for disjunction
5. Rules for negation

Contradiction

expressions of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$

denoted by $\perp$

they let you derive anything

bottom-elimination

Rules for natural deduction

1. Rules for conjunction
2. Rules for double negation
3. Rules for implication
4. Rules for disjunction
5. Rules for negation

Examples

$\neg p \vee q \vdash p \rightarrow q$

Examples

$\neg p \vee q \vdash p \to q$

$p \to q, p \to \neg q \vdash \neg p$

Examples

$\neg p \vee q \vdash p \to q$

$p \to q, p \to \neg q \vdash \neg p$

$p \to \neg p \vdash \neg p$

Recall our first example

- If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late. *Therefore*, there were taxis at the station.

- If p and not q, then r. Not r. p. *Therefore*, q.

- $p \wedge \neg q \to r, \neg r, p \vdash q$

Derived rules

- Modus Tollens (MT)

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- proof by contradiction

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- Modus Tollens (MT)
- $\neg\neg$i
- proof by contradiction
- law of excluded middle ($\phi \vee \neg\phi$ is true)

Examples

Use LEM to show the validity of $p \to q \vdash \neg p \lor q$

Provable equivalence

- $\phi \dashv\vdash \psi$
- $p \wedge q \to r \dashv\vdash p \to (q \to r)$
- $p \wedge q \to p \dashv\vdash r \vee \neg r$

Thank you!