COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 19 & 20 (Normal forms, Herbrand's Theorem)

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Prenex Normal Form

A predicate logic formula is in prenex normal form if it written as a string of quantifiers and bound variables, called the prefix, followed by a quantifier-free part, called the matrix.

$Q_1 y_1 Q_2 y_2 \dots Q_n y_n \ F$

where $Q_i \in \{\forall, \exists\}$, $n \geq 0$, and F contains no quantifiers.

Every formula is equivalent to a formula in prenex normal form.

Example

$\neg(\exists x\ P(x, y) \vee \forall z\ Q(z)) \wedge \exists w\ Q(w)$

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F,

then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Renaming bound variables

Let F denote the formula $Qx\ G$ where Q is a quantifier. Let y be a variable that does not occur in G.

Then $F \equiv Qy\ (G[y/x])$.

Rectified formulas

A formula is rectified if no variable occurs both bound and free and if all quantifiers in the formula refer to different variables.

You can rectify a formula by renaming bound variables.

Example: $\forall x\ \exists y\ P(x, f(y))\ \wedge\ \forall y\ (Q(x, y) \vee R(x))$

Every formula is equivalent to a rectified formula in prenex form.

Skolem Form

We say that a rectified prenex formula is in Skolem form if it does not contain any occurrence of the existential quantifier.

A rectified prenex formula can be transformed to an equisatisfiable formula in Skolem form by using extra function symbols.

$\forall x\ \exists y\ P(x, y)$ and $\forall x\ P(x, f(x))$ are equisatisfiable.

Skolem Form

Let $F = \forall y_1 \; \forall y_2 \; \dots \; \forall y_n \; \exists z \; G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F, write

$$F' = \forall y_1 \; \forall y_2 \; \dots \; \forall y_n \; G[f(y_1, y_2, \dots, y_n)/z].$$

Then F and F' are equisatisfiable.

Example

$$\forall x \; \exists y \; \forall z \; \exists w \; (\neg P(a, w) \vee Q(f(x), y))$$

Proof: Renaming bound variables

Let F denote the formula $Qx\ G$ where Q is a quantifier. Let y be a variable that does not occur in G.

Then $F \equiv Qy\ (G[y/x])$.

Proof:

Proof: Skolem Form

Let $F = \forall y_1 \; \forall y_2 \; \ldots \; \forall y_n \; \exists z \; G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F, write

$$F' = \forall y_1 \; \forall y_2 \; \ldots \; \forall y_n \; G[f(y_1, y_2, \ldots, y_n)/z].$$

Then F and F' are equisatisfiable.

Proof:

Proof: Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F,

then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Proof: reading exercise

Herbrand structure

universe is the set of ground terms

terms and function symbols being interpreted "as themselves"

built from syntax

Herbrand structure

Definition 1. Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is called a *Herbrand structure* if the following hold:

1. The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
2. For every constant symbol c in σ we have $c_{\mathcal{H}} = c$.
3. For every k-ary function symbol f in σ and for all ground terms $t_1, t_2 \ldots, t_n \in U_{\mathcal{H}}$ we have $f_{\mathcal{H}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$.

Interpretation of a ground term

Let $\mathcal{H}$ be a Herbrand structure, and t be a ground term.

Then, $\mathcal{H}[\![t]\!] = t$.

Translation Lemma for Herbrand structures

Let $\mathcal{H}$ be a Herbrand structure, F be a formula, and t be a ground term.

Then $\mathcal{H} \models F[t/x]$ if and only if $\mathcal{H}_{[x \mapsto t]} \models F$.

Herbrand's Theorem and Proof

Let $F := \forall x_1 \ldots \forall x_n\ F^*$ be a closed formula in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

Proof:

Example

Is the following formula satisfiable?

$$F := \exists x_1 \exists x_2 \exists x_3 \ (\neg(\neg P(x_1) \rightarrow P(x_2)) \wedge \neg(\neg P(x_1) \rightarrow \neg P(x_3)))$$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol
- does not work for $\forall x_1 \exists x_2 \; F^*$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol
- does not work for $\forall x_1 \exists x_2 \; F^*$
- the presence of a function symbols in its Skolem form makes each Herbrand structure infinite

Thank you!