COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 21 & 22 (Ground Resolution, Undecidability results)

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Herbrand expansion

Let $F := \forall x_1 \ldots \forall x_n\ F^*$ be a closed formula in Skolem form with matrix F^*.

$E(F) := \{F^*[t_1/x_1] \ldots [t_n/x_n] \mid t_1 \ldots t_n$ are ground terms $\}$

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A closed formula F in Skolem form is satisfiable iff $E(F)$ is satisfiable when considered as a set of propositional formulas.

Proof:

Ground resolution

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$E(F)$ is unsat iff some finite subset of $E(F)$ is unsat. (Compactness theorem)

Ground resolution

A closed formula F in Skolem form is unsatisfiable iff there is a propositional resolution proof of $\square$ from $E(F)$.

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Soundness and completeness of propositional resolution says that we can derive $\square$ from $E(F)$ using resolution.

Generalized version of Ground Resolution Theorem

Let $F_1, F_2, \ldots, F_n$ be closed formulas in Skolem form

whose respective matrices $F_1^*, F_2^*, \ldots, F_n^*$ are in CNF.

$F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is unsatisfiable iff there is a propositional resolution proof of $\square$ from the ground instances[1] of clauses from $F_1^*, F_2^*, \ldots, F_n^*$.

[1] a ground instance of F is a formula obtained by replacing all variables in F with ground terms

Example

Let us use ground resolution to show that (a), (b), and (c) together entail (d).

(a) Everyone in the class is either sleepy, bored, or day-dreaming.

(b) All those who are bored are sleepy.

(c) Someone in the class is not day-dreaming.

(d) Someone in the class is sleepy.

Example

Show that $\forall x\ \exists y\ (P(x) \to Q(y)) \to \exists y\ \forall x\ (P(x) \to Q(y))$ is a valid sentence.

Compactness

- Compactness of sets of ground formulas – A set of ground quantifier-free formulas has a model iff every finite subset of it has a model.
- Compactness of closed formulas – A set of first-order sentences has a model iff every finite subset of it has a model.
- Löwenheim Skolem Theorem – If a set of closed formulas has a model, then it has a model with a domain (universe) which is at most countable.

Semi-decidability of validity

Validity of first-order formulas is semi-decidable[2].

Proof:

[2]a semi-decision procedure for validity should return “valid” if a valid formula is given as input, but otherwise may compute forever

Semi-decidability of validity

Validity of first-order formulas is semi-decidable[2].

Proof:

Semi-Decision Procedure for Validity
Input: Closed formula F
Output: Either that F is valid or compute forever

Compute a Skolem-form formula G equisatisfiable with $\neg F$
Let $G_1, G_2, \ldots$ be an enumeration of the Herbrand expansion $E(G)$
for $n = 1$ to ∞ **do**
begin
  if $\square \in \text{Res}^*(G_1 \cup \ldots \cup G_n)$ **then** stop and output "F is valid"
end

[2] a semi-decision procedure for validity should return "valid" if a valid formula is given as input, but otherwise may compute forever

Let us try this on the formula

$\exists x \forall y \ P(x, y) \rightarrow \forall y \exists x \ P(x, y)$

Undecidability results

Post's Correspondence Problem (PCP) is undecidable.

Undecidability of validity follows from undecidability of PCP.

Since F is unsatisfiable iff $\neg F$ is valid, satisfiability must also be undecidable.

Satisfiability is not even semi-decidable (because, for any F, either F is valid or $\neg F$ is satisfiable).

Proof

Reference material:

`https://www.cs.ox.ac.uk/people/james.worrell/lecture13-2015.pdf`

Closed formula for a general PCP instance

Given a general instance $\mathbf{P} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ of PCP we have the formulas

$$
\begin{aligned}
F_1 &= \bigwedge_{i=1}^{k} P(f_{x_i}(e), f_{y_i}(e)) \\
F_2 &= \forall u \, \forall v \bigwedge_{i=1}^{k} (P(u, v) \rightarrow P(f_{x_i}(u), f_{y_i}(v))) \\
F_3 &= \exists u \, P(u, u) \, .
\end{aligned}
$$

The PCP instance **P** has a solution iff $F_1 \wedge F_2 \rightarrow F_3$ is valid.

Thank you!