COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 5-6 (Normal forms, Propositional Resolution)

Kumar Madhukar

madhukar@cse.iitd.ac.in

August 10th and 14th, 2023

Semantic equivalence, Satisfiability, Validity

For propositional logic formulas ϕ and ψ, we say that they are semantically equivalent (denoted as $\phi \equiv \psi$) iff $\phi \vDash \psi$ and $\psi \vDash \phi$ hold.

ϕ is said to be valid if $\vDash \phi$ (tautologies are exactly the valid formulas)

ϕ is said to be satisfiable if it has a valuation in which it evaluates to true.

Semantic equivalence, Satisfiability, Validity

For propositional logic formulas ϕ and ψ, we say that they are semantically equivalent (denoted as $\phi \equiv \psi$) iff $\phi \vDash \psi$ and $\psi \vDash \phi$ hold.

ϕ is said to be valid if $\vDash \phi$ (tautologies are exactly the valid formulas)

ϕ is said to be satisfiable if it has a valuation in which it evaluates to true.

ϕ is satisfiable iff $\neg\phi$ is not valid.

Distributivity and De Morgan's Laws

Normal Forms

Negation Normal Form (NNF)

A well-formed formula (wff) is in NNF if it uses only $\vee$, $\wedge$, and *literals*.

- Every wff is logically equivalent to a wff in NNF.
- Exercise: convert $\neg(p \rightarrow (p \wedge q))$ into NNF.

Normal Forms

Disjunctive Normal Form (DNF)

A well-formed formula (wff) is in DNF if it is a disjunction of one or more terms, where each term is a conjunction of one or more literals.

Note: p, $(p \wedge q \wedge \neg r)$, and $(p \vee q)$ are all in DNF.

- Every wff is logically equivalent to a wff in DNF.
  How? From the truth-table, or using logical equivalences.

- Exercise: convert $\neg(p \rightarrow (p \wedge q))$ into DNF.

Normal Forms

Conjunctive Normal Form (CNF)

A well-formed formula (wff) is in CNF if it is a conjunction of one or more terms, where each term is a disjunction of one or more literals.

- Every wff is logically equivalent to a wff in CNF.
  How? From the truth-table, or using logical equivalences.

- Exercise: convert $\neg(p \rightarrow (p \wedge q))$ into CNF.

From truth tables

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Figure 1: Truth table for F

Why care about CNF formulas?

validity checking is easy (it otherwise takes time exponential in the no. of atoms)

consider $(\neg q \vee p \vee r) \wedge (\neg p \vee r) \wedge q$

$\models (\neg q \vee p \vee r) \wedge (\neg p \vee r) \wedge q$ holds iff

$\models (\neg q \vee p \vee r)$, $\models (\neg p \vee r)$, $\models q$ all three hold

but that is easy to check:
a disjunction of literals is valid iff they have a pair of complementary literals

Propositional Resolution

- set representation of CNF formulas
- proof rule: $(\neg p \lor \phi), (p \lor \psi)$ resolve to give $(\phi \lor \psi)$
- derivation of $\Box$ gives a refuation
- refutation is the way proofs are done
- e.g. $(x \lor \neg y), (y \lor z), (\neg x \lor \neg y \lor z) \vdash z$

  is proved by deriving $\Box$ from $\{\{x, \neg y\}, \{y, z\}, \{\neg x, \neg y, z\}, \{\neg z\}\}$ using resolution

Resolution Lemma

Let F be a CNF formula represented as a set of clauses. Suppose R is a resolvent of two clauses C_1 and C_2 in F, then $F \equiv F \cup \{R\}$.

Soundness

If there is a derivation of $\Box$ from F then F is unsatisfiable.

Completeness

If F is unsatisfiable then there is a derivation of $\Box$ from F.

Lecture notes on Resolution

https://www.cs.ox.ac.uk/people/james.worrell/lec6-2015.pdf

Thank you!