COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 7-8 (Hilbert's Axiomatization, Soundness and Completeness, Compactness)

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Hilbert's proof system

Axioms and inference rule

Derivations

A derivation of α is a finite sequence of formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that:

- $\beta_n = \alpha$
- each β_i is either an instance of one of the axioms, or modus ponens applied to β_j and β_k such that $j, k < i$.

Example

derivation of $\alpha \to \alpha$

Soundness and Completeness

for all formulas α, $\vdash \alpha$ *iff* $\models \alpha$

Soundness is easy to prove

Completeness

We need to prove that if α is a tautology, then α is derivable.

Suppose we prove that $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$ (TODO)
Then, it suffices to prove that if α is not derivable, then α is not a tautology.

Completeness

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Suppose we prove that $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$ (TODO)
Then, it suffices to prove that if α is not derivable, then α is not a tautology.

Suppose we prove that $\neg\neg\alpha \to \alpha$. (TODO)

Then, if α is not derivable, $\neg\neg\alpha$ is also not derivable.

Completeness

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Suppose we prove that $\neg\neg\alpha \to \alpha$. (TODO)

Then, if α is not derivable, $\neg\neg\alpha$ is also not derivable.

Let us call α consistent if $\nvdash \neg\alpha$.

So, $\neg\alpha$ is consistent. Suppose we prove that if β is consistent then β is satisfiable. (TODO).

Completeness

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Suppose we prove that $\neg\neg\alpha \to \alpha$. (TODO)

Then, if α is not derivable, $\neg\neg\alpha$ is also not derivable.

Let us call α consistent if $\nvdash \neg\alpha$.

So, $\neg\alpha$ is consistent. Suppose we prove that if β is consistent then β is satisfiable. (TODO).

So, $\neg\alpha$ is satisifiable. Therefore, α is not valid (it is not a tautology).

Deduction Theorem

For a set of formulas Γ, and formulas α, β, $\quad \Gamma \cup \{\alpha\} \vdash \beta \quad \text{iff} \quad \Gamma \vdash \alpha \to \beta$.

Our TODO list

- $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
- $\neg\neg\alpha \to \alpha$
- For all formulas β, if β is consistent then β is satisfiable.

Completeness Proof

β consistent $\quad \rightarrow \quad \beta$ satisfiable

- every consistent set can be extended to a maximal consistent set (MCS)
- let X be an MCS; for all formulas α, $\quad v_X \models \alpha$ *iff* $\alpha \in X$
  (where v_X is the valuation that every atomic proposition in X to *true*)

Derivability and Logical Consequence

Strong Completeness

Let $X \subseteq \Phi$ and $\alpha \in \Phi$. Then $X \vDash \alpha$ *iff* $X \vdash \alpha$.

Compactness Theorem

Let $X \subseteq \Phi$ and $\alpha \in \Phi$. Then $X \vDash \alpha$ *iff* there exists $Y \subseteq_{fin} X, Y \vDash \alpha$.

Finite Satisfiability

Let $X \subseteq \Phi$. Then, X is satisfiable *iff* every $Y \subseteq_{fin} X$ is satisfiable.

Proof of Compactness Theorem

Proof of Strong Completeness

(left as an exercise)

Thank you!