COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 13 & 14 (Predicate Logic)

Kumar Madhukar

madhukar@cse.iitd.ac.in

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The need for a richer language

- the logical aspects of natural and artificial languages are much richer
  - than what propositional logic can capture
  - limited to sentence components like not, and, or, if ... then
- consider the following declarative sentence

  Every student is younger than some instructor.
- a propositional atom denoting this fails to capture the finer logical structure of this sentence.

The need for a richer language

Every student is younger than some instructor.

this is about being a student, being an instructor, and being younger than somebody else

we would like a mechanism to express these with their logical relationships

this is what we will use predicates for

Predicates

- we can use predicates S, I, and Y

- $S(\text{John})$ – John is a student
- $I(\text{Paul})$ – Paul is an instructor
- $Y(\text{John}, \text{Paul})$ – John is younger than Paul

- the meaning of these symbols must be specified exactly

Predicates

- can we now express "Every student is younger than some instructor."

Predicates

- can we now express "Every student is younger than some instructor."
- we don't want to write down every instance of S(.)
- use variables as place holders for concrete values
- $S(x)$ – x is a student
- $I(x)$ – x is an instructor
- $Y(x, y)$ – x is younger than y

Quantifiers

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- can we now express "Every student is younger than some instructor."
- Every student is younger than some instructor.
- $\forall$ (for all) and $\exists$ (there exists)
- the quantifiers always come attached to a variable
- $\forall x$ (for all x) and $\exists z$ (there exists z)

Example

- can we now express "Every student is younger than some instructor."

Example

- can we now express "Every student is younger than some instructor."
- $\forall x \; (S(x) \rightarrow (\exists y \; (I(y) \wedge Y(x, y))))$
- for every x, if x is a student, then there is some y such that y is an instructor and x is younger than y
- predicates can have any (finite) number of arguments (arity)

Another example

- Not all birds can fly.
- $B(x)$ – x is a bird
- $F(x)$ – x can fly

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- alternatively, $\exists x\ (B(x) \wedge \neg F(x))$

Another example

- Not all birds can fly.
- $B(x)$ – x is a bird
- $F(x)$ – x can fly
- $\neg(\forall x \ (B(x) \rightarrow F(x)))$
- alternatively, $\exists x \ (B(x) \wedge \neg F(x))$
- does this formula evaluate to `true` in the world we currently live in?

Another example

Every child is younger than its mother.

Another example

Every child is younger than its mother.

$$\forall x \; \forall y \; ((C(x) \land M(x, y)) \to Y(x, y))$$

Another example

Andy and Paul have the same maternal grandmother.

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Andy and Paul have the same maternal grandmother.

$$\forall x \; \forall y \; \forall u \; \forall v \; ((M(x, y) \wedge M(y, Andy) \wedge M(u, v) \wedge M(v, Paul)) \rightarrow x = u)$$

Another example

Andy and Paul have the same maternal grandmother.

$$\forall x\ \forall y\ \forall u\ \forall v\ ((M(x, y) \land M(y, Andy) \land M(u, v) \land M(v, Paul)) \rightarrow x = u)$$

- function symbols can help us avoid the inelegent encoding
- equality has been used as a special predicate

As a formal language

there are two sorts of things involved in a predicate logic formula:

objects that we are talking about – constants, variables, $m(a)$, $g(x, y)$

expressions denoting objects are called terms

the other sort of things are formulas

Terms

Formulas

Binding properties

- $\neg$, $\forall y$, $\exists y$ bind most tightly
- then $\vee$ and $\wedge$
- then $\rightarrow$, which is right-associative

Another example

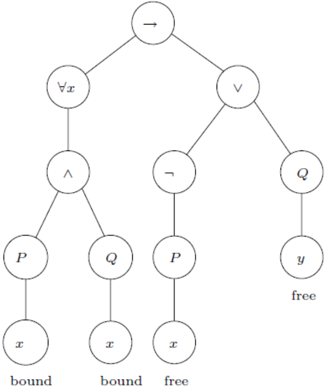
Every son of my father is my brother.

Parse trees

$\forall x\ ((P(x) \to Q(x)) \land S(x, y))$

Free and bound variables

$(\forall x\ (P(x) \land Q(x))) \to (\neg P(x) \lor Q(y))$



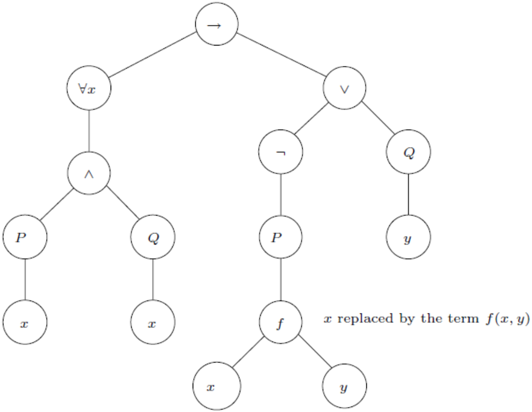
Substitution

given a variable x, a term t, and a formula ϕ

we define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t

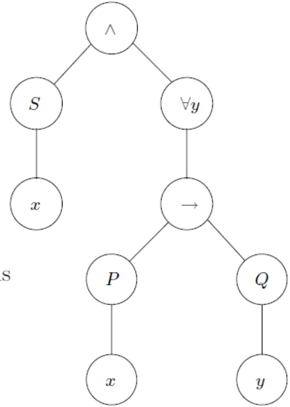
Substitution: example

example: $((\forall x\ (P(x) \land Q(x))) \to (\neg P(x) \lor Q(y)))[f(x,y)/x]$



Undesired side-effects of substitution

substitution must be avoided if t is not free for x in ϕ



the term $f(y, y)$ is not free for x in this formula

Proof theory (natural deduction rules)

(from the book by Huth and Ryan, pages 107-117)

Quantifier Equivalence

(from the book by Huth and Ryan, pages 117-122)

Thank you!