COL703: Logic for Computer Science (Jul-Nov 2023)

Lectures 15 & 16 (Semantics of Predicate Logic)

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Semantics

$$\forall x \forall y \ (P(x, y) \rightarrow \exists z \ (\neg(z = x) \land \neg(z = y) \land P(x, z) \land P(z, y)))$$

- suppose variables take values from real numbers
- P(x, y) represents $x < y$

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$$\forall x \forall y \; (P(x, y) \rightarrow \exists z \; (\neg(z = x) \wedge \neg(z = y) \wedge P(x, z) \wedge P(z, y)))$$

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- suppose variables take values from natural numbers
- P(x, y) represents $x < y$

σ-structure[1]

Given a signature σ, a *σ-structure* (or *assignment*) $\mathcal{A}$ consists of:

- a non-empty set $U_\mathcal{A}$ called the *universe* of the structure;
- for each k-ary predicate symbol P in σ, a k-ary relation $P_\mathcal{A} \subseteq \underbrace{U_\mathcal{A} \times \cdots \times U_\mathcal{A}}_{k}$;
- for each k-ary function symbol symbol f in σ, a k-ary function, $f_\mathcal{A} : \underbrace{U_\mathcal{A} \times \cdots \times U_\mathcal{A}}_{k} \to U_\mathcal{A}$;
- for each constant symbol c, an element $c_\mathcal{A}$ of $U_\mathcal{A}$;
- for each variable x an element $x_\mathcal{A}$ of $U_\mathcal{A}$.

[1] `https://www.cs.ox.ac.uk/people/james.worrell/lecture9-2015.pdf`

Evaluating terms

- For a constant symbol c we define $\mathcal{A}[\![c]\!] \stackrel{\text{def}}{=} c_{\mathcal{A}}$.
- For a variable x we define $\mathcal{A}[\![x]\!] \stackrel{\text{def}}{=} x_{\mathcal{A}}$.
- For a term $f(t_1, \ldots, t_k)$, where f is a k-ary function symbol and $t_1, \ldots, t_k$ are terms, we define $\mathcal{A}[\![f(t_1, \ldots, t_k)]\!] \stackrel{\text{def}}{=} f_{\mathcal{A}}(\mathcal{A}[\![t_1]\!], \ldots, \mathcal{A}[\![t_k]\!])$.

Satisfaction relation

1. $\mathcal{A} \models P(t_1, \ldots, t_k)$ if and only if $(\mathcal{A}[\![t_1]\!], \ldots, \mathcal{A}[\![t_k]\!]) \in P_{\mathcal{A}}$.
2. $\mathcal{A} \models (F \land G)$ if and only if $\mathcal{A} \models F$ and $\mathcal{A} \models G$.
3. $\mathcal{A} \models (F \lor G)$ if and only if $\mathcal{A} \models F$ or $\mathcal{A} \models G$.
4. $\mathcal{A} \models \neg F$ if and only if $\mathcal{A} \not\models F$.
5. $\mathcal{A} \models \exists x\, F$ if and only if there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \models F$.
6. $\mathcal{A} \models \forall x\, F$ if and only if $\mathcal{A}_{[x \mapsto a]} \models F$ for all $a \in U_{\mathcal{A}}$.

If we are working in first-order logic with equality then we additionally have

7. $\mathcal{A} \models t_1 = t_2$ if and only if $\mathcal{A}[\![t_1]\!] = \mathcal{A}[\![t_2]\!]$.

Some definitions

- quantifier-depth of a formula

  atomic formulas have 0 qd; $\text{qd}(\neg\phi) = \text{qd}(\phi)$;
  $\text{qd}(\phi \vee \psi) = \text{qd}(\phi \wedge \psi) = \mathit{max}(\text{qd}(\phi), \text{qd}(\psi))$
  $\text{qd}(\exists x\ \phi) = \text{qd}(\forall x\ \phi) = 1 + \text{qd}(\phi)$

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- ground terms – variable-free terms

- a closed formula or a sentence – formula with no free variables

  e.g. $\forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$ is closed
  $\forall x\ (x < (y+1))$ is not closed

Examples

- undirected graph as a σ-structure (σ containing one binary relation E), with the interpretation of E as the edge relation
- σ with one binary relation $<$, interpreted in the usual way over integers, rationals, reals

More definitions

- a first-order formula F over σ is satisfiable if there is a σ-structure $\mathcal{A}$ such that $\mathcal{A} \models F$
- if F is not satisfiable, it is called unsatisfiable
- F is called valid if (and only if) $\neg F$ is unsatisfiable
- for a set of formulas $\mathcal{S}$, we say $\mathcal{S} \models F$ to mean that every σ-structure $\mathcal{A}$ that satisfies $\mathcal{S}$ also satisfies F

Exercise

Consider a signature σ with a constant symbol 0, a unary function symbol s, and a unary predicate symbol P.

Is $P(0) \wedge \forall x \, (P(x) \rightarrow P(s(x))) \wedge \exists x \, \neg P(x)$ satisfiable?

Relevance Lemma

Suppose $\mathcal{A}$ and $\mathcal{A}'$ are σ-assignments with the same universe, and identical interpretation of the predicate, function, and constant symbols in σ.

If $\mathcal{A}$ and $\mathcal{A}'$ give the same interpretation to each variable occurring free in some σ-formula F,

then $\mathcal{A} \models F$ iff $\mathcal{A}' \models F$.

Proof: Induction.

Special case

If F is a closed formula (or a sentence), and $\mathcal{A}$ and $\mathcal{A}'$ are assignments that only differ in interpretation of variables,

then $\mathcal{A} \models F$ iff $\mathcal{A}' \models F$.

Logical equivalence

First-order formulas F and G are logically equivalent, denoted $F \equiv G$, if for all σ-assignments $\mathcal{A}$, we have $\mathcal{A} \models F$ iff $\mathcal{A} \models G$.

Example

$$\neg\forall x\ F \ \equiv\ \exists x \neg F$$

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$$(\forall x\ F \wedge G) \ \equiv\ \forall x\ (F \wedge G) \quad \text{(if } x \text{ does not occur free in } G\text{)}$$

Exercises

- $\exists x \ (A(x) \rightarrow B(x)) \quad \leftrightarrow \quad \forall x \ A(x) \rightarrow \exists x \ B(x)$

- $\forall x \ (A(x) \vee B(x)) \quad \rightarrow \quad \forall x \ A(x) \vee \exists x \ B(x)$

Thank you!